ELLIPTIC BETA INTEGRALS, SUPERCONFORMAL INDICES, AND THE YANG-BAXTER EQUATION

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ELLIPTIC HYPERGEOMETRIC INTEGRALS
a new class of special functions (V.S., 2000, 2003)

Univariate case: contour integrals

$$I = \int_C \Delta(u) du$$

are EHIs, if $\Delta(u)$ satisfies a first order finite difference equation

$$\Delta(u + \omega_1) = h(u; \omega_2, \omega_3)\Delta(u),$$

where $h(u; \omega_2, \omega_3)$ is an elliptic function,

$$h(u + \omega_2) = h(u + \omega_3) = h(u), \quad \mathrm{Im}(\omega_2/\omega_3) \neq 0.$$

Given incommensurate $\omega_{1,2,3} \in \mathbb{C}$ define the bases

$$p = e^{2\pi i \omega_3/\omega_2}, \quad q = e^{2\pi i \omega_1/\omega_2}.$$

Let $\Delta(u) := \rho(z)$ be meromorphic in $z = e^{2\pi i u/\omega_2}$:

$$I = \int \rho(z) \frac{dz}{z}, \qquad \rho(qz) = h(z; p)\rho(z), \quad h(pz) = h(z),$$

$$h(z) = \prod_{k=1}^{m} \frac{\theta(t_k z; p)}{\theta(w_k z; p)}, \qquad \prod_{k=1}^{m} t_k = \prod_{k=1}^{m} w_k,$$

$$\theta(z; p) = (z; p)_\infty (p z^{-1}; p)_\infty, \qquad (z; p)_\infty = \prod_{j=0}^{\infty} (1 - z p^j).$$

Due to the factorization, sufficient to solve

$$f(qz) = \theta(z;p)f(z).$$

A particular solution: the elliptic gamma function

$$\Gamma(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}p^{j+1}q^{k+1}}{1 - zp^j q^k}, \qquad |p|, |q| < 1.$$

Hence,

$$I = \int \prod_{k=1}^{m} \frac{\Gamma(t_k z;p,q)}{\Gamma(w_k z;p,q)} \frac{dz}{z}, \qquad \prod_{k=1}^{m} t_k = \prod_{k=1}^{m} w_k.$$

Elliptic hypergeometric series = sums of residues of particular sequences of poles. Conventions

$$\Gamma(t_1,\ldots,t_k;p,q) := \Gamma(t_1;p,q)\cdots\Gamma(t_k;p,q),$$
$$\Gamma(tz^{\pm 1};p,q) := \Gamma(tz;p,q)\Gamma(tz^{-1};p,q).$$

THE ELLIPTIC BETA INTEGRAL

Theorem (V.S., 2000). Let $|p|, |q|, |t_j| < 1$, $\prod_{j=1}^{6} t_j = pq$, $\mathbb{T} =$ the unit circle. Then

$$\frac{(p;p)_\infty (q;q)_\infty}{4\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\prod_{j=1}^{6} \Gamma(t_j z^{\pm 1};p,q)}{\Gamma(z^{\pm 2};p,q)} \frac{dz}{z} = \prod_{1 \le j < k \le 6} \Gamma(t_j t_k;p,q).$$

This was a principally new exactly computable integral:

– an elliptic analogue of Newton's binomial theorem

– a top known generalization of Euler's beta integral

– obeys $W(E_6)$ group of symmetries

– many multidimensional extensions to integrals on root systems

ELLIPTIC FOURIER TRANSFORMATION

Definition of integral Bailey pairs (V.S., 2003)

$$\beta(w,t) = M(t)_{wz}\alpha(z,t)$$

$$= \frac{(p;p)_\infty (q;q)_\infty}{4\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\Gamma(tw^{\pm 1}z^{\pm 1};p,q)}{\Gamma(t^2, z^{\pm 2};p,q)} \alpha(z,t) \frac{dz}{z}.$$

Integral Bailey lemma:

$$\alpha'(w,st) = D(s;u,w)\alpha(w,t), \qquad D(s;u,w)D(s^{-1};u,w) = 1,$$

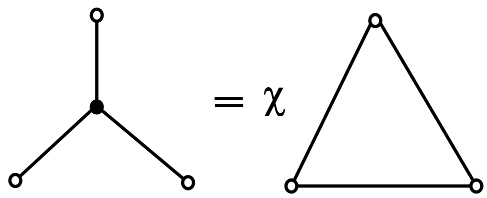
$$D(s;u,w) := \Gamma(\sqrt{pq}s^{-1}u^{\pm 1}w^{\pm 1};p,q),$$

$$\beta'(w,st) = D(t^{-1};u,w)M(s)_{wx}D(st;u,x)\beta(x,t).$$

From $\beta'(w,st) = M(st)_{wz}\alpha'(z,st) \Rightarrow$

$$M(s)_{wx}D(st;u,x)M(t)_{xz} = D(t;u,w)M(st)_{wz}D(s;u,z).$$

This is an operator form of the elliptic beta integral known as the star-triangle relation



Inversion relation $t \to t^{-1}$: (V.S., Warnaar, 2005)

$$M(t^{-1})_{wz} M(t)_{zx} f(x) = f(w).$$

Inversion = sign change (like in the Fourier transformation)

SUPERCONFORMAL INDEX

Four-dimensional (!) $\mathcal{N}=1$ SUSY gauge field theory:
$G_{full} = SU(2,2|1) \times G \times F$

$J_i, \overline{J}_i$ ($SU(2)$ subgroup generators, or Lorentz rotations),
$P_\mu, Q_\alpha, \overline{Q}_{\dot\alpha}$ (supertranslations),
$K_\mu, S_\alpha, \overline{S}_{\dot\alpha}$ (special superconformal transformations),
H (dilations) and R ($U(1)_R$-rotations).
Internal symmetries: gauge group G^a, flavor symmetry F_k

For $Q = \overline{Q}_1$ and $Q^\dagger = -\overline{S}_1$, one has $Q^2 = (Q^\dagger)^2 = 0$ and

$$\{Q, Q^\dagger\} = 2\mathcal{H}, \qquad \mathcal{H} = H - 2\overline{J}_3 - 3R/2$$

The superconformal index: (KMMR, Romelsberger, 2005)

$$I(y;p,q) = \mathrm{Tr}\Big((-1)^{\mathrm{F}} p^{\mathcal{R}/2+J_3} q^{\mathcal{R}/2-J_3} e^{\sum_k f_k F_k} e^{-\beta\mathcal{H}}\Big),$$

$\mathcal{R} = H - R/2$ and F is the fermion number,
$p, q, y_k = e^{f_k}, e^{-\beta}$ are group parameters (fugacities).

It counts BPS states $\mathcal{H}|\psi\rangle = 0$ or cohomology of Q, $Q^\dagger$ operators (hence, no β-dependence).

"Physical" (not rigorous) computation yields a matrix integral:

$$I(y;p,q) \ = \ \int_G d\mu(z)\, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{ind}\big(p^n, q^n, z^n, y^n\big)\right)$$

with the Haar measure $d\mu(z)$ and single particle states index

$$\mathrm{ind}(p,q,z,y) = \frac{2pq-p-q}{(1-p)(1-q)}\chi_{adj_G}(z) + \sum_j \frac{(pq)^{R_j/2}\chi_{r_F,j}(y)\chi_{r_G,j}(z) - (pq)^{1-R_j/2}\chi_{\bar{r}_F,j}(y)\chi_{\bar{r}_G,j}(z)}{(1-p)(1-q)}.$$

$\chi_{R_F,j}(y)$ and $\chi_{R_G,j}(z)$ are characters of the respective representations, and R_j are the R-charges.

For the unitary group $SU(N)$, $z=(z_1,\ldots,z_N)$, $\prod_{a=1}^N z_a = 1$,

$$\int_{SU(N)} d\mu(z) \ = \ \frac{1}{N!}\int_{\mathbb{T}^{N-1}} \Delta(z)\Delta(z^{-1})\prod_{a=1}^{N-1}\frac{dz_a}{2\pi \mathrm{i} z_a},$$

$$\Delta(z) \ = \ \prod_{1\leq a<b\leq N}(z_a-z_b), \qquad \text{the Vandermonde determinant.}$$

Where is the elliptic beta integral here ?

The left-hand side: $G = SU(2)$, $F = SU(6)$, representations

1) vector superfield: $(adj, 1)$,

$$\chi_{SU(2),adj}(z) = z^2 + z^{-2} + 1,$$

2) chiral superfield: (f, f),

$$\chi_{SU(2),f}(z) = z + z^{-1}, \qquad R_f = 1/3,$$

$$\chi_{SU(6),f}(y) = \sum_{k=1}^{6} y_k, \quad \chi_{SU(6),\bar{f}}(y) = \sum_{k=1}^{6} y_k^{-1}, \quad \prod_{k=1}^{6} y_k = 1,$$

and $t_k = (pq)^{1/6} y_k$, $k = 1, \ldots, 6$

The right-hand side: $G = 1$, $F = SU(6)$ with the single chiral superfield $T_A: \quad \Phi_{ij} = -\Phi_{ji}$,

$$\chi_{SU(6),T_A}(y) = \sum_{1 \le i < j \le 6} y_i y_j, \qquad R_{T_A} = 2/3.$$

A Wess-Zumino type theory for the confined colored particles.

The elliptic beta integral describes the confinement phenomenon in the simplest $4d$ supersymmetric quantum chromodynamics!

Seiberg, 1994; Dolan-Osborn, 2008.

The process of integrals' computation = transition from UV (weak coupling) to IR (strong coupling) physics.

EHIs = new matrix models

EHIs = new computable path integrals in $4d$ QFT

Symmetries of EHIs = general Seiberg dualities. Very many new identities, dualities, other results (V.S., Vartanov, 2008-2012; GPRRY, 2009-2012)

SOLUTION OF THE YANG-BAXTER EQUATION
(joint work with S. Derkachov, 2012)

The Yang Baxter equation

$$\mathbb{R}_{12}(u-v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{23}(v)=\mathbb{R}_{23}(v)\,\mathbb{R}_{13}(u)\,\mathbb{R}_{12}(u-v)$$

$\mathbb{R}_{jk}$ acts in $\mathbb{V}_j\otimes\mathbb{V}_k\subset\mathbb{V}_1\otimes\mathbb{V}_2\otimes\mathbb{V}_3$

Baxter's (1972) 8-vertex model solution: $\dim\mathbb{V}_j=2$,

$$\mathbb{R}^B_{12}(u)=\sum_{a=0}^{3}w_a(u)\,\sigma_a\otimes\sigma_a\quad;\quad w_a(u)=\frac{\theta_{a+1}(u+\eta|\tau)}{\theta_{a+1}(\eta|\tau)}\,,$$

Sklyanin's (1982) extension: $\dim\mathbb{V}_1=\dim\mathbb{V}_2=2,\dim\mathbb{V}_3=\infty$,

$$\mathrm{R}_{13}(u)=\mathrm{L}_{13}(u):=\sum_{a=0}^{3}w_a(u)\,\sigma_a\otimes\mathbf{S}^a$$

$$=\begin{pmatrix} w_0(u)\,\mathbf{S}^0+w_3(u)\,\mathbf{S}^3 & w_1(u)\,\mathbf{S}^1-iw_2(u)\,\mathbf{S}^2\\ w_1(u)\,\mathbf{S}^1+iw_2(u)\,\mathbf{S}^2 & w_0(u)\,\mathbf{S}^0-w_3(u)\,\mathbf{S}^3\end{pmatrix}.$$

$$\mathbb{R}^B_{12}(u-v)\,\mathrm{L}_{13}(u)\,\mathrm{L}_{23}(v)=\mathrm{L}_{23}(v)\,\mathrm{L}_{13}(u)\,\mathbb{R}^B_{12}(u-v)\qquad\Rightarrow$$

$$\mathbf{S}^\alpha\,\mathbf{S}^\beta - \mathbf{S}^\beta\,\mathbf{S}^\alpha = i\cdot\left(\mathbf{S}^0\,\mathbf{S}^\gamma + \mathbf{S}^\gamma\,\mathbf{S}^0\right)\ ,$$

$$\mathbf{S}^0\,\mathbf{S}^\alpha - \mathbf{S}^\alpha\,\mathbf{S}^0 = i\,\mathbf{J}_{\beta\gamma}\cdot\left(\mathbf{S}^\beta\,\mathbf{S}^\gamma + \mathbf{S}^\gamma\,\mathbf{S}^\beta\right)\ ,$$

$(\alpha,\beta,\gamma) =$ a cycle of $(1,2,3)$, $\mathbf{J}_{12} = \frac{\theta_1^2(\eta)\theta_4^2(\eta)}{\theta_2^2(\eta)\theta_3^2(\eta)}$, etc.
Explicit realization as finite-difference operators Sklyanin, 1983

$$[\mathbf{S}^a\,\Phi]\,(z) = \frac{\mathrm{i}^{\delta_{a,2}}\theta_{a+1}(\eta)}{\theta_1(2z)}\Big[\,\theta_{a+1}\,(2z-2\eta\ell)\cdot\Phi(z+\eta) - \theta_{a+1}\,(-2z-2\eta\ell)\cdot\Phi(z-\eta)\Big]\,,$$

where ℓ is the *spin*.

GENERAL CONSTRUCTION FOR $\dim\mathbb{V}_j = \infty$

- Take a defining RLL-relation for $\dim\mathbb{V}_1 = \dim\mathbb{V}_2 = \infty$, $\dim\mathbb{V}_3 = 2$ in the form

$$\mathbb{R}^?_{12}(u-v)\,\sigma_3\mathrm{L}_1(u)\,\sigma_3\,\mathrm{L}_2(v) = \sigma_3\,\mathrm{L}_2(v)\,\sigma_3\mathrm{L}_1(u)\,\mathbb{R}^?_{12}(u-v)\,.$$

  $\mathrm{L}_1 = \mathrm{L}$ with z replaced by z_1, $\mathrm{L}_2 = \mathrm{L}$ with z replaced by z_2. Solve it using permutation operators S_j, $j = 1,2,3$, generating $\mathfrak{S}_4$;
- prove that the resulting $\mathbb{R}_{12}(u)$-operator obeys the general YBE for $\dim\mathbb{V}_j = \infty$.

Extract the permutation operator $\mathbb{R}_{12}(u) := \mathbb{P}_{12}\,\mathrm{R}_{12}(u) \Rightarrow$

$$\mathrm{R}_{12}(u{-}v)\,\mathrm{L}_1(u_1,u_2)\,\sigma_3\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_1(v_1,v_2)\,\sigma_3\,\mathrm{L}_2(u_1,u_2)\,\mathrm{R}_{12}(u{-}v)\,,$$

$$u_1 = \frac{u}{2} + \eta\,(\ell_1 + \frac{1}{2})\ ,\ u_2 = \frac{u}{2} - \eta\,(\ell_1 + \frac{1}{2})\ \ ;$$
$$v_1 = \frac{v}{2} + \eta\,(\ell_2 + \frac{1}{2})\ ,\ v_2 = \frac{v}{2} - \eta\,(\ell_2 + \frac{1}{2})$$

Notice that $\mathrm{R}_{12}(u-v) \equiv \mathrm{R}_{12}(u_1,u_2|v_1,v_2)$ permutes parameters in the product of L-operators

$$\mathbf{u} \equiv (u_1,u_2,v_1,v_2), \quad s\mathbf{u} = (v_1,v_2,u_1,u_2), \qquad s = s_2s_1s_3s_2,$$
$$s_1\mathbf{u} = (u_2,u_1,v_1,v_2)\ ,\ s_2\mathbf{u} = (u_1,v_1,u_1,v_2)\ ,\ s_3\mathbf{u} = (u_1,u_2,v_2,v_1),$$

$\Rightarrow$ Definig relations for S_j-operators,

$$\mathrm{S}_1(\mathbf{u})\,\mathrm{L}_1(u_1,u_2) = \mathrm{L}_1(u_2,u_1)\,\mathrm{S}_1(\mathbf{u})\ ;\ \mathrm{S}_3(\mathbf{u})\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_2(v_2,v_1)\,\mathrm{S}_3(\mathbf{u}),$$
$$\mathrm{S}_2(\mathbf{u})\,\mathrm{L}_1(u_1,u_2)\,\sigma_3\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_1(u_1,v_1)\,\sigma_3\,\mathrm{L}_2(u_2,v_2)\,\mathrm{S}_2(\mathbf{u})\,.$$

Coxeter relations:

$$\mathrm{S}_i^2 = 1,\ \mathrm{S}_i\mathrm{S}_j = \mathrm{S}_j\mathrm{S}_i,\ |i-j| > 1,\ \mathrm{S}_j\mathrm{S}_{j+1}\mathrm{S}_j = \mathrm{S}_{j+1}\mathrm{S}_j\mathrm{S}_{j+1}$$

Theorem.

$$\mathrm{R}_{12}^{?}(\mathbf{u}) = \mathrm{S}_2(s_1s_3s_2\mathbf{u})\,\mathrm{S}_1(s_3s_2\mathbf{u})\,\mathrm{S}_3(s_2\mathbf{u})\,\mathrm{S}_2(\mathbf{u}).$$

solves the initial RLL=LLR relation.

Permutation of parameters in the product of **three** L-operators

$$\mathrm{L}_1(u_1,u_2)\,\sigma_3\,\mathrm{L}_2(v_1,v_2)\,\sigma_3\,\mathrm{L}_3(w_1,w_2)$$
$$\longrightarrow \mathrm{L}_1(w_1,w_2)\,\sigma_3\,\mathrm{L}_2(v_1,v_2)\,\sigma_3\,\mathrm{L}_3(u_1,u_2)$$

is realized in two different ways ⇒

$$\mathrm{R}_{12}(v_1,v_2|w_1,w_2)\,\mathrm{R}_{23}(u_1,u_2|w_1,w_2)\,\mathrm{R}_{12}(u_1,u_2|v_1,v_2)$$
$$= \mathrm{R}_{23}(u_1,u_2|v_1,v_2)\,\mathrm{R}_{12}(u_1,u_2|w_1,w_2)\,\mathrm{R}_{23}(v_1,v_2|w_1,w_2)\,.$$

A miracle:

Denote $\quad p = e^{2\pi \mathrm{i}\tau}, \qquad q = e^{4\pi \mathrm{i}\eta}$

Then, for a special choice of periodic factors,

$$\mathrm{S}_2(z_1,z_2,\mathbf{u}) = D(e^{2\pi\mathrm{i}(v_1-u_2)};x_2,x_1),\;\; x_2 = e^{2\pi\mathrm{i}z_2},\;\; x_1 = e^{2\pi\mathrm{i}z_1},$$

$$\mathrm{S}_1 = e^{-\pi\mathrm{i}z_2^2/\eta} M(e^{-2\pi\mathrm{i}(u_1-u_2)})_{x_1x_2} e^{\pi\mathrm{i}z_1^2/\eta}, \quad \mathrm{S}_3 = \mathrm{S}_1(z_1 \leftrightarrow z_2).$$

where $D(...,x_1)$ and $M_{x_1x_2}$ are the Bailey lemma entries !
The Coxeter relations = identities for D and M

Ell. beta integral = star-triangle rel. = Coxeter rel.

Similarity transformation ⇒ an R-operator

$$[\mathbb{R}_{12}(\mathbf{u})f](x_1,x_2) = \frac{(p;p)_\infty^2 (q;q)_\infty^2}{(4\pi \mathrm{i})^2} \Gamma(\sqrt{pq} x_1^{\pm 1} x_2^{\pm 1} e^{2\pi \mathrm{i}(v_2-u_1)}; p,q)$$
$$\times \int_{\mathbb{T}^2} \frac{\Gamma(e^{2\pi \mathrm{i}(v_1-u_1)} x_2^{\pm 1} x^{\pm 1}, e^{2\pi \mathrm{i}(v_2-u_2)} x_1^{\pm 1} y^{\pm 1}; p,q)}{\Gamma(e^{4\pi \mathrm{i}(v_1-u_1)}, e^{4\pi \mathrm{i}(v_2-u_2)}, x^{\pm 2}, y^{\pm 2}; p,q)}$$
$$\times \Gamma(\sqrt{pq} e^{2\pi \mathrm{i}(v_1-u_2)} x^{\pm 1} y^{\pm 1}; p,q) f(x,y) \frac{dx}{x} \frac{dy}{y}.$$

Symmetric in p and q $\Rightarrow$ there is second RLL-relation !

$$\mathbb{R}_{12}(u-v)\, \sigma_3 \mathrm{L}_1'(u)\, \sigma_3\, \mathrm{L}_2'(v) = \sigma_3\, \mathrm{L}_2(v)'\, \sigma_3 \mathrm{L}_1(u)'\, \mathbb{R}_{12}(u-v)\,,$$

$\mathrm{L}'(u) = \mathrm{L}(\text{fixed } u,\ \text{fixed } g = \eta(2\ell+1),\ 2\eta \leftrightarrow \tau)$.

$\Rightarrow$ A second copy of the Sklyanin algebra:

THE ELLIPTIC MODULAR DOUBLE (V.S., 2008)

$$\tilde{\mathbf{S}}^a(z,\ell) = \mathrm{e}^{\frac{2\pi i}{\tau} z^2} \cdot \frac{(i)^{\delta_{a,2}} \theta_{a+1}(\frac{\tau}{2})|2\eta)}{\theta_1(2z|2\eta)} \Big[\, \theta_{a+1}\left(2z - g + \frac{\tau}{2}|2\eta\right) \cdot \mathrm{e}^{\frac{1}{2}\tau\partial}$$
$$- \theta_{a+1}\left(-2z - g + \frac{\tau}{2}|2\eta\right) \cdot \mathrm{e}^{-\frac{1}{2}\tau\partial}\, \Big] \cdot \mathrm{e}^{-\frac{2\pi i}{\tau} z^2}\,.$$

Existence of EMD + meromorphy in $e^{2\pi \mathrm{i} z}$ $\Rightarrow$ removal of periodic factors $\Rightarrow$ uniqueness.

No limit $p \to 0$ ($\mathrm{Im}(\tau) \to +\infty$), but $\exists$ 2nd EMD which in this limit goes to Faddeev's (1999) modular double.

A $4d/2d$-correspondence.

$4d$ superconformal indices describe partition functions of integrable models of $2d$ spin systems (V.S., 2010)

Act by the Coxeter/Bailey relation on a localized spin ⇒

$$\int_0^{2\pi} \rho(u) W(\xi-\alpha; x, u) W(\alpha+\gamma; y, u) W(\xi-\gamma; w, u) du$$
$$= \chi W(\alpha; y, w) W(\xi-\alpha-\gamma; x, w) W(\gamma; x, y), \qquad (1)$$

where

$$W(\alpha; x, y) = \Gamma(e^{-\alpha} e^{\mathrm{i}(\pm x \pm y)}; p, q)$$
$$\rho(u) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi} \theta(e^{2\mathrm{i}u}; p) \theta(e^{-2\mathrm{i}u}; q),$$
$$\chi = \Gamma(e^{-\alpha}, e^{-\gamma}, e^{\alpha+\gamma-\xi}; p, q), \qquad e^{-\xi} = \sqrt{pq}.$$

A functional star-triangle relation (Bazhanov, Sergeev, 2010). Ising type models: vertices carry spins $z, w, \ldots$, edges carry Boltzmann weights W, integration (summation) over z-spin values.

EHIs symmetry transformations = star-star relations (V.S., 2010)

Seiberg duality = Kramers-Wannier type duality transformations for elementary partition functions of $2d$ spin systems

$4d \to 3d$ reduction ⇒ a new Faddeev-Volkov type YBE/STR solution (V.S., 2010)

$$W(\alpha; x, z) = \gamma^{(2)}(\alpha - \eta \pm \mathrm{i}x \pm \mathrm{i}z; \omega),$$

where the modified q-gamma function (noncompact quantum dilogarithm, hyperbolic gamma function, etc)

$$\gamma^{(2)}(u;\omega_1,\omega_2) \propto \frac{(e^{2\pi \mathrm{i} u/\omega_1}\tilde{q};\tilde{q})_\infty}{(e^{2\pi \mathrm{i} u/\omega_2};q)_\infty}.$$

With $x,z \to x+\mu, z+\mu,\ \mu \to \infty \Rightarrow$ the original Faddeev-Volkov solution (1995) of STR

$$W(\alpha;x,z) = \gamma^{(2)}(\alpha - \eta \pm \mathrm{i}(x-z);\omega)$$

Full partition functions = SCIs of quiver gauge theories

CONCLUSION

Elliptic hypergeometric functions are universal objects with wide applications.

In mathematics: analytic theory of finite-difference equations (e.g., elliptic hypergeometric equation), harmonic analysis on root systems, representation theory, theory of automorphic forms, approximation theory, continued fractions, combinatorics, topology, etc

In theoretical physics: $4d$ supersymmetric dualities, integrable N-particle quantum mechanical systems, topological field theories, $2d$ solvable models of statistical mechanics and noncomact spin chains, random matrices and stochastic determinantal processes, etc